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On a Trigonometric Inequality of Vinogradov

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The sum $f(m, n) = \sum_{a=1}^{m-1} (|\sin \pi an/m|/|\sin \pi a/m|)$ arises in bounding incomplete exponential sums. In this article we show that for positive integers m, n with $m > 1$, $f(m, n) < (4/\pi^2)\, m \log m + 0.38m + 0.608 + 0.116\, d^2/m$, where $d = (m, n)$. This improves earlier bounds for $f(m, n)$. The constant $4/\pi^2$ in the main term is shown to be best possible. © 1987 Academic Press, Inc.

Let m and n be positive integers with $m > 1$. The sum

$$f(m, n) = \sum_{a=1}^{m-1} \frac{|\sin \pi an/m|}{|\sin \pi a/m|}$$

has arisen many times in the literature. In particular, it arises in estimating "incomplete" exponential sums of the form $\sum_{x=b+1}^{b+n} e_m(g(x))$, where $g(x)$ is a polynomial with integer coefficients, $b \in \mathbb{Z}$, and $e_m(y) = e^{(2\pi i/m)y}$. Vinogradov [11, Ex. 12, Chap. 5] observed that this "incomplete" sum could be expressed as a fraction of the "complete" exponential sum $\sum_{x=0}^{m-1} e_m(g(x))$ plus an error term

$$\sum_{x=b+1}^{b+n} e_m(g(x)) = \frac{n}{m} \sum_{x=0}^{m-1} e_m(g(x)) + \theta\, \Delta f(m, n), \tag{1.1}$$

where $|\theta| < 1$ and $\Delta = \max_{y=1,\dots,m-1} |\sum_{x=0}^{m-1} e_m(g(x) + xy)|$. He [11, Ex. 11, Chap. 3] obtained the estimate for $f(m, n)$,

$$f(m, n) < \begin{cases} m \log m - \dfrac{m}{3} \log\left(2\left[\dfrac{m}{6}\right] + 1\right) & \text{for } m \geqslant 6 \\ m \log m - \dfrac{m}{2} & \text{for } m \geqslant 12 \\ m \log m - m & \text{for } m \geqslant 60. \end{cases} \tag{1.2}$$

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As an application of (1.1) and (1.2) Vinogradov [11 Ex. 12, Chap. 5] showed that if p is a prime then the number of quadratic residues (mod p) among the integers $b+1, b+2, \ldots, b+n$ is $(n/2)+(\theta/2)\sqrt{p}\log p$, where $|\theta|<1$. The bound (1.2) for $f(m, n)$ has also been used by Chalk [1], Chalk and Williams [2], Smith [9], and Spackman [10] in solving questions concerning the distribution of solutions of polynomial congruences. Mordell [5, 6] and Serre [8] have made use of the more crude bound $f(m, n) < m \log m$ for similar problems.

Lidl and Neiderreiter [4, Lemma 8.80] obtained the improved bound $f(m, n) < (2/\pi) m \log m + \frac{2}{5}m + n$, and used if for proving results on the distribution of points in linear recurrences over finite fields.

We shall prove the sharper inequality:

THEOREM 1. *For any positive integers m, n with $m>1$ we have*

$$f(m, n) < \frac{4}{\pi^2} m \log m + 0.38m + 0.608 + 0.116\frac{d^2}{m},$$

where $d=(m, n)$. In particular, if $m>8$ then $f(m, n) < m \log m - m$.

We also prove a mean value property for $f(m, n)$; specifically:

THEOREM 2. *The function $f(m, n)$ has the following properties.*

(i) $m^{-1}\sum_{n=1}^{m} f(m, n) = (4/\pi^2)\, m \log m + (4/\pi^2)(\gamma - \log(\pi/2))\, m + O(\log m \log(\log m))$, *where γ is Euler's constant*, $\gamma = 0.57721\ldots$.

(ii) $\sum_{n=1}^{m} (f(m, n) - (4/\pi^2)\, m \log m)^2 \ll m^3 \log m$.

It follows from Theorem 2 that the constant $4/\pi^2$ in Theorem 1 is best possible. However, the constant 0.38 in the second term of Theorem 1 is by no means best possible. One would hope that 0.38 could be replaced by a value very close to the value of the constant in the second term of Theorem 2(i), which is roughly 0.0507. The fact that $f(m, n) < m \log m - m$ for $m>8$ follows immediately for $m>12$ from the first part of Theorem 1 and can be checked by direct computation for $m=9, 10, 11, 12$. When $m=8$ this inequality fails, indeed $f(8, 3) > 8 \log 8 - 8$.

We wish to thank Professor Hugh L. Montgomery for pointing out to us that the constant $4/\pi^2$ in the main term of Theorem 1 could be obtained and for the helpful discussions on proving this result. We also wish to thank Professor Paul T. Bateman for many helpful comments on simplifying the proofs and sharpening the constants, and Professor D. J. Lewis for his comments on earlier drafts of this article.

2. Proof of Theorem 1

To start, we note the following consequence of the Euler–Maclaurin summation formula found for example in Gradshteyn and Ryzhik [3, 0.131],

$$0 < \log m + \gamma + \frac{1}{2m} - \sum_{v=1}^{m} \frac{1}{v} \ll \frac{1}{m^2} \tag{2.1}$$

The following three lemmas are needed in the proof of Theorem 1.

Lemma 2.1. *For any integer $m \geqslant 2$ and real number x,*

$$\sum_{a=1}^{m-1} \frac{|\sin ax|}{a} < \frac{2}{\pi} (\log m + \gamma + \log 2) + \frac{3}{\pi m}.$$

Proof. This is essentially the result of problem 38, part VI of Polya–Szego [7], except we make their proof more precise to obtain an extra savings. Using the Fourier series

$$|\sin \theta| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{v=1}^{\infty} \frac{\cos 2v\theta}{4v^2 - 1}$$

we have

$$\sum_{a=1}^{m-1} \frac{|\sin ax|}{a} = \frac{2}{\pi} \sum_{v=1}^{m-1} \frac{1}{v} - \frac{4}{\pi} \sum_{v=1}^{\infty} \frac{B(m-1, 2vx)}{4v^2 - 1},$$

where $B(m, x) = \sum_{k=1}^{m} k^{-1} \cos(kx)$ for $m \in \mathbb{Z}^+$, $x \in \mathbb{R}$. If m is odd then $B(m, x)$ takes on its minimum value when $x = \pi$, as shown in Polya–Szego [7, No. 27, Part VI], and so for all $x \in \mathbb{R}$,

$$B(m, x) \geqslant -1 + \frac{1}{2} - \frac{1}{3} + \cdots - \frac{1}{m} > -\log 2 - \frac{1}{m+1}.$$

Thus, if m is even

$$\begin{aligned} \sum_{a=1}^{m-1} \frac{|\sin ax|}{a} &< \frac{2}{\pi} \sum_{v=1}^{m-1} \frac{1}{v} + \frac{4}{\pi} \left(\log 2 + \frac{1}{m} \right) \sum_{v=1}^{\infty} \frac{1}{4v^2 - 1} \\ &= \frac{2}{\pi} \sum_{v=1}^{m-1} \frac{1}{v} + \frac{2}{\pi} \left(\log 2 + \frac{1}{m} \right), \end{aligned}$$

and consequently if m is odd,

$$\sum_{a=1}^{m-1} \frac{|\sin ax|}{a} < \frac{2}{\pi} \sum_{v=1}^{m} \frac{1}{v} + \frac{2}{\pi}\left(\log 2 + \frac{1}{m+1}\right).$$

The result of the lemma now follows from (2.1).

LEMMA 2.2. *For* $0 < x < 1$ *we have*

$$\frac{1}{\sin \pi x} < \frac{1}{\pi}\left(\frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x}\right).$$

Proof. We make use of the well known expansion from complex analysis,

$$\begin{aligned}\frac{1}{\sin \pi x} &= \frac{1}{\pi} \sum_{k \in Z} \frac{(-1)^k}{x-k} = \frac{1}{\pi} \lim_{m \to \infty} \sum_{m}^{m} \frac{(-1)^k}{x-k} \qquad (x \notin \mathbb{Z})\\ &= \frac{1}{\pi}\left(\frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x}\right) + \frac{1}{\pi} \sum_{k=2}^{\infty} \left(\frac{(-1)^k}{x-k} + \frac{(-1)^k}{x+k}\right)\\ &= \frac{1}{\pi}\left(\frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x}\right) - \frac{2x}{\pi} \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 - x^2}.\end{aligned}$$

The infinite sum in the preceding equation is clearly positive for $0 < x < 1$ and so the lemma follows.

LEMMA 2.3. *For any integers* n, m *with* $m > 1$ *we have*

$$\frac{2m}{\pi} - \frac{2}{\pi}(4 - \pi)\frac{d^2}{m} < \sum_{a=1}^{m} |\sin \pi an/m| < \frac{2m}{\pi} - \frac{\pi d^2}{6m},$$

where $d = (n, m)$.

Proof. Suppose first that $(n, m) = 1$. Then we have

$$\sum_{a=1}^{m} |\sin \pi an/m| = \sum_{a=1}^{m} |\sin \pi a/m| = \sum_{a=1}^{m} \sin(\pi a/m) = \cot\left(\frac{\pi}{2m}\right).$$

Now for $0 < x \leqslant \pi/4$ we have

$$\cot x - \frac{1}{x} = -2x \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2 - x^2} < -2x \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} = -\frac{x}{3},$$

and

$$\cot x - \frac{1}{x} > -2x \sum_{k=1}^{\infty} \frac{1}{(k^2 - (1/16))\,\pi^2} = -\frac{4x}{\pi^2} \sum_{k=1}^{\infty} \left(\frac{1}{k-(1/4)} - \frac{1}{k+(1/4)} \right)$$

$$= -\frac{16}{\pi^2} x \left(1 - \frac{\pi}{4}\right).$$

Therefore for $m > 1$,

$$\frac{2m}{\pi} - \frac{2}{\pi}(4-\pi)\frac{1}{m} < \cot\left(\frac{\pi}{2m}\right) < \frac{2m}{\pi} - \frac{\pi}{6m}, \tag{2.2}$$

which proves the lemma when $(n, m) = 1$.

Suppose now that $(n, m) = d \geqslant 1$. Then we have

$$\sum_{a=1}^{m} \left|\sin\left(\frac{\pi a n}{m}\right)\right| = \sum_{a=1}^{m} \left|\sin\left(\frac{\pi a n'}{m'}\right)\right|,$$

where $n' = n/d$, $m' = m/d$ and $(n', m') = 1$. As a runs from 1 to m, $n'a$ runs through a complete set of residues (mod m') d times, so that

$$\sum_{a=1}^{m} \left|\sin\left(\frac{\pi a n}{m}\right)\right| = d \sum_{a=1}^{m'} \left|\sin\left(\frac{\pi a n'}{m'}\right)\right| = d \cot\left(\frac{\pi}{2m'}\right).$$

The lemma follows again from (2.2).

Proof of Theorem 1. Let m, n be positive integers with $m > 1$. By Lemma 2.2 we see that

$$f(m, n) < \frac{m}{\pi} \sum_{a=1}^{m-1} \frac{|\sin \pi a n/m|}{a} + \frac{m}{\pi} \sum_{a=1}^{m-1} \frac{|\sin \pi a n/m|}{m-a} - \frac{m}{\pi} A, \tag{2.3}$$

where A denotes the sum

$$A = \sum_{a=1}^{m-1} \frac{|\sin \pi a n/m|}{m+a}.$$

The first two sums on the right-hand side of (2.3) are equal in value and bounded above by the value given in Lemma 2.1.

To bound A from below we observe that

$$2A = \sum_{a=1}^{m-1} \left|\sin \frac{\pi a n}{m}\right| \left(\frac{1}{m+a} + \frac{1}{2m-a}\right)$$

$$= \sum_{a=1}^{m-1} \left|\sin \frac{\pi a n}{m}\right| \frac{3m}{(m+a)(2m-a)}.$$

By the inequality of the arithmetic and geometric means we have

$$2A > \sum_{a=1}^{m-1}{}' \left|\sin\frac{\pi an}{m}\right| \frac{3m}{(3m/2)^2} = \frac{4}{3m}\sum_{a=1}^{m-1}{}' \left|\sin\frac{\pi an}{m}\right|,$$

and so by Lemma 2.3 we conclude that

$$A > \frac{4}{3\pi} - \frac{4}{3\pi}(4-\pi)\frac{d^2}{m^2} \tag{2.4}$$

where $d = (m, n)$.

Combining (2.3) and (2.4) we see that

$$\begin{aligned} f(m,n) < {} & \frac{2m}{\pi}\left(\frac{2}{\pi}\log m + \frac{2}{\pi}(\gamma + \log 2) + \frac{3}{\pi m}\right) \\ & - \frac{4m}{3\pi^2}\left(1 - (4-\pi)\frac{d^2}{m^2}\right), \end{aligned}$$

completing the proof of the theorem.

3. Proof of Theorem 2

Lemma 3.1. *For $m \geqslant 2$ we have*

$$\sum_{a=1}^{m-1}{}' \frac{1}{\sin \pi a/m} = \frac{2m}{\pi}\left(\log m + \gamma - \log\frac{\pi}{2}\right) + O(1).$$

Proof. We start with the expansion

$$\sum_{a=1}^{m-1}{}' \frac{1}{\sin \pi a/m} = \frac{1}{\pi}\sum_{a=1}^{m-1}{}' \sum_{k\in Z} \frac{(-1)^k}{(a/m)-k} = \frac{m}{\pi}\sum_{a=1}^{m-1}{}' \sum_{k\in Z}\frac{(-1)^k}{a-mk}.$$

By pairing the $(-k)$th and $(k+1)$st terms in the preceding sum we obtain

$$\begin{aligned} \sum_{a=1}^{m-1}{}' \frac{1}{\sin(\pi a/m)} &= \frac{m}{\pi}\sum_{a=1}^{m-1}{}' \sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{a+mk} - \frac{1}{a-m(k+1)}\right) \\ &= \frac{2m}{\pi}\sum_{k=0}^{\infty}(-1)^k\sum_{a=1}^{m-1}{}' \frac{1}{mk+a} \\ &= \frac{2m}{\pi}\sum_{a=1}^{m-1}{}' \frac{1}{a} + \frac{2m}{\pi}\sum_{k=1}^{\infty}(-1)^k\sum_{a=1}^{m-1}{}' \frac{1}{mk+a}. \end{aligned} \tag{3.2}$$

Now, by (2.1),

$$\sum_{a=1}^{m-1}{}' \frac{1}{mk+a} = \log\left(1+\frac{1}{k}\right) - \frac{1}{m(k+1)} + m^{-1}O(k^{-2}),$$

and so

$$\begin{aligned}
&\sum_{k=1}^{\infty} (-1)^k \sum_{a=1}^{m-1}{}' \frac{1}{mk+a} \\
&\quad = \sum_{k=1}^{\infty} (-1)^k \log\left(1+\frac{1}{k}\right) - m^{-1} \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1} + m^{-1}O\left(\sum_{k=1}^{\infty} k^{-2}\right) \\
&\quad = -\log\left(\frac{\pi}{2}\right) + O\left(\frac{1}{m}\right),
\end{aligned} \tag{3.3}$$

by Wallis' formula for $\pi/2$. Combining (3.2) and (3.3) and using (2.1) yields the result of the lemma.

Proof of Theorem 2. (i) By Lemma 2.3 we have

$$\begin{aligned}
\sum_{n=1}^{m} f(m,n) &= \sum_{a=1}^{m-1}{}' \frac{1}{\sin \pi a/m} \sum_{n=1}^{m} |\sin \pi an/m| \\
&= \frac{2m}{\pi} \sum_{a=1}^{m-1}{}' \frac{1}{\sin \pi a/m} + \frac{1}{m} O\left(\sum_{a=1}^{m-1}{}' \frac{(a,m)^2}{\sin \pi a/m}\right).
\end{aligned} \tag{3.4}$$

Now,

$$\begin{aligned}
\sum_{a=1}^{m-1}{}' \frac{(a,m)^2}{\sin \pi a/m} &\leqslant 2 \sum_{a=1}^{[(m-1)/2]} \frac{(a,m)^2}{\sin \pi a/m} + \frac{m^2}{4} \\
&\leqslant m \sum_{a=1}^{[(m-1)/2]} \frac{(a,m)^2}{a} + \frac{m^2}{4} \\
&< m \sum_{a=1}^{m} \frac{(a,m)^2}{a} \\
&= m \sum_{d|m} \sum_{\substack{a=1 \\ (a,m)=d}}^{m} \frac{d^2}{a} \\
&= m \sum_{d|m} d \sum_{\substack{b=1 \\ (b,m/d)=1}}^{m/d} \frac{1}{b} \\
&\ll m \sum_{d|m} d \log m \\
&\ll m^2 \log m \log\log m.
\end{aligned} \tag{3.5}$$

From (3.4), (3.5) and Lemma 3.1 we conclude that

$$\sum_{n=1}^{m} f(m,n) = \frac{4m^2}{\pi^2}\left(\log m + \gamma - \log\frac{\pi}{2}\right) + O(m) + \frac{1}{m}\,O(m^2 \log m \log\log m),$$

which finishes the proof of part (i). We remark that it seems difficult to avoid the $\log\log m$ factor in (3.5).

(ii) Set $B = (4/\pi^2)\,\mathrm{m}\log m$. Then from Theorems 1 and 2(i) we have that $f(m,n) < B + O(m)$ and that $\sum_{n=1}^{m} f(m,n) = mB + O(m^2)$. Thus we see that

$$\begin{aligned}\sum_{n=1}^{m} (f(m,n)-B)^2 &= \sum_{n=1}^{m} f(m,n)^2 - 2B\sum_{n=1}^{m} f(m,n) + mB^2\\ &< m(B+O(m))^2 - 2B(mB + O(m^2)) + mB^2\\ &= O(m^2 B) = O(m^3 \log m).\end{aligned}$$

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